LPTENS-96/52
hep-th@xxx/9611012

Duality and BPS spectra in N=2 supersymmetric QCD

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I review, with some pedagogy, two different approaches to the computation of BPS spectra in $N = 2$ supersymmetric QCD with gauge group SU(2). The first one is semiclassical and has been widely used in the literature. The second one makes use of constraints coming from the non perturbative, global structure of the Coulomb branch of these theories. The second method allows for a description of discontinuities in the BPS spectra at strong coupling, and should lead to accurate tests of duality conjectures in $N = 2$ theories.

1. General overview

The understanding of strong-weak coupling dualities, or S dualities, in supersymmetric gauge theories (on which I will focus) and superstring theory, is certainly the most outstanding problem in high energy physics at present. Typically, S duality relates a theory A with coupling constant g to another theory B with coupling constant $1/g$. Testing a S duality conjecture thus requires the knowledge of some non perturbative informations, at least for one of the theory. In some cases, one expects that A and B coincide, i.e. they have the same lagrangian, with different parameters. Such "self-dual" field theories are certainly the most fascinating one from the theoretical point of view. The $N = 4$ theory is strongly believed to belong to this class, as well as other, more interesting and more difficult to study, $N = 2$ theories. In other cases, the duality is more subtle, and can only be valid in the low energy limit, as in the asymptotically free theories.

A complete proof of the duality properties in these theories seems out of reach at present. However, some accurate tests can be done. One of them is to look at the low energy effective actions of the theories conjectured to be dual, and check whether a sensible duality transformation can be found relating them. This approach strongly suggests that the $N = 2$ theory obtained from the $N = 4$ theory by adding a term $m \operatorname{tr} \Phi^2$ to the superpotential, as well as the $N = 2$ theory with four flavours of quarks in the fundamental representation of the gauge group (for SU(2)), should be self-dual [1, 2].

Another test is to look at the BPS spectra. This test is more accurate than simply looking at the low energy effective action, since it directly probes the Hilbert space of states. When the spectrum is stable, one can deduce strong coupling results from weak coupling, semiclassical, calculations. For instance, this was done in [3, 4, 5] for the SU(2) $N = 4$ theory, with results in agreement with S duality, as we will see below. However, when the number of supersymmetries decreases down to 2, there may exist curves in moduli space, which generically separate a strong coupling from a weak coupling region, across which the BPS spectrum is discontinuous [1, 2]. In these cases, the semiclassical method is hopeless, since it is only valid at weak coupling. One must devise another method to understand the jumping phenomenon, then compute the strong coupling spectrum and check if it is compatible with S duality. Note that in the $N = 4$ theory, complete $SL(2, \mathbf{Z})$ invariance requires that all the BPS states (n_e, n_m), where n_e and n_m are relatively prime integers corresponding respectively to the electric and magnetic charges, must exist as quantum stable states in the theory [3]. This is no longer the case when some discontinuity curves are present: if a duality transformation relates two region in the param-

eter space which are separated by such a curve, we can only say that the spectrum in one region must be the dual of the spectrum in the other region.

In this short lecture, I will describe how the strong coupling BPS spectra have been computed in the asymptotically free theories, following [6, 7]. These theories are not self-dual, though electric-magnetic duality still plays a profound rôle [1, 2]. They provide an example where the "duality" group of the low energy effective action (or of the spectral curve associated with it), typically a subgroup of finite index of $\mathrm{SL}(2,\mathbf{Z})$, has nothing to do with the duality transformations which can be thought as being valid quantum mechanically. This provides the first steps toward the study of the expected self-dual field theories. I also give a short introduction to the semiclassical quantization, which in any case provides useful informations, in particular concerning the quantum numbers carried by the solitonic states.

2. The semiclassical quantization

In the bosonic sector, the classical study of the BPS monopole configurations amounts to solving the Bogomol'nyi equation [8]

$$B = \pm \mathrm{D}\phi, \tag{1}$$

where B is the (non abelian) magnetic field and ϕ the Higgs scalar transforming in the adjoint representation of the gauge group SU(2). The configuration of ϕ at infinity gives an element of $\pi_2(S^2) = \mathbf{Z}$, which is the magnetic charge n_m. At fixed n_m, the set of solutions of (1) can be parametrized by $4n_m$ real parameters. For $n_m = 1$, these are the position of the center of mass of the soliton, as well as a fourth periodic collective coordinate corresponding to the electric charge. For n_m widely separated monopoles of charge 1, which correspond to a configuration of global charge n_m, we will have 4 parameters describing the global motion and electric charge, and $4n_m - 4$ parameters describing the relative motions and electric charges. The $4n_m$ dimensional parameter (moduli) space $\mathcal{M}_{n_m}$ was studied in great detail in [9], and has nice mathematical properties. In particular, it is a hyperkähler manifold, as we will see below. The knowledge of the explicit form of the metric on $\mathcal{M}_{n_m}$ allows one to study the classical low energy dynamics of a n_m-monopole configuration, which corresponds to geodesic motion [10]. This can be done in full generality when $n_m = 2$, which is the only case where the metric is known [9] (see [11] and references therein for recent developments).

However, the study of the classical motion is not enough for our purposes. What we need is to investigate the *quantum* stability of multi-monopole BPS configurations. This requires to look at the quantum mechanics associated with the classical dynamical system describing the geodesic motions, and look for stable bound states satisfying the Bogomol'nyi bound. Since $\dim \mathcal{M}_{n_m} = 4n_m$, this quantum mechanics has $4n_m$ bosonic degrees of freedom z^α. Because of supersymmetry, one would expect to have in addition $4n_m$ fermionic collective coordinates λ^α. These indeed come from the zero modes of the Dirac equation associated with the adjoint fermions, whose number can be computed using Callias' index theorem [12]. The action then reads [13]

$$S = \int \mathrm{d}t\, G_{\alpha\beta}(z)(\dot z^\alpha \dot z^\beta + i\lambda^\alpha D_t \lambda^\beta). \tag{2}$$

$G_{\alpha\beta}$ is the metric on $\mathcal{M}_{n_m}$, D_t the covariant derivative associated with it, and $\dot z = \mathrm{d}z/\mathrm{d}t$. As the original theory has two supersymmetries in four space-time dimensions, which correspond to eight real supersymmetry generators, and as a BPS configuration breaks half of the supersymmetries, S must have four real supersymmetries. This is possible if and only if the target space $\mathcal{M}_{n_m}$ is hyperkähler [14]. Note that this nice mathematical property of $\mathcal{M}_{n_m}$ can be proved independently [9]. If N_f matter hypermultiplets are also present (we will limit ourselves to zero bare masses), we will have $2n_m N_f$ additional fermionic zero modes κ^{jA}, $1 \le j \le 2N_f$, $1 \le A \le n_m$. The action will then pick up a new term [15, 16]

$$S_{\mathrm{m}} = \int \mathrm{d}t\, \Big(i\kappa^{jA}\mathcal{D}_t\kappa^{jA} + \frac{1}{2}F^{AB}_{\alpha\beta}\lambda^\alpha\lambda^\beta\kappa^{jA}\kappa^{jB}\Big), \tag{3}$$

where $\mathcal{D}_t$ is the covariant derivative corresponding to a natural $\mathrm{O}(n_m)$ connexion related to the isospinor fermionic zero modes, and F the associated curvature two-form. The total action still has four real supersymmetries in spite of the mismatch between the number of fermionic and bosonic zero modes, supersymmetry being non linearly realized. The standard quantization procedure leads to

$$\{\lambda^\alpha, \lambda^\beta\} = \delta^{\alpha\beta};\ \{\kappa^{jA}, \kappa^{lB}\} = \delta^{jl}\delta^{AB}, \tag{4}$$

which are Clifford algebras whose representation theory is well known. The hamiltonian H associated with the action $S + S_{\mathrm{m}}$ is the square of a Dirac operator coupled to the $\mathrm{O}(n_m)$ connexion. Its normalizable zero modes will correspond to quantum mechanically stable BPS states (note that only the zero modes will correspond to states saturating the Bogomol'nyi bound). Finding these zero modes is a very hard mathematical problem which requires the knowledge of the metric on $\mathcal{M}_{n_m}$ and the use of advanced index theory. This is why only very partial results have been obtained up to now in this direction [16]. There exists however a trick, introduced and applied with some success by Porrati in [4] for the $N = 4$ theory. Since H is a supersymmetric hamiltonian, the existence of zero modes is equivalent to the fact that supersymmetry is not broken in the supersymmetric quantum mechanics. A very convenient way of proving that susy is not broken is then to compute the Witten index [17]. The main drawback of this method is that it does not allow to count the number of the zero modes, and thus leads only to a partial computation of the BPS spectrum. It also requires the use of not so well established results concerning the asymptotic behaviour of the multi-monopole moduli space $\mathcal{M}_{n_m}$.

We will not pursue this route here. But before presenting a completely different method in the next Section, let us briefly discuss the quantum numbers carried by the BPS states. From the representation theory of the algebras (4) we know that the wave function of any BPS state can be written

$$|\Psi\rangle = f(z^\alpha)\,|\psi_0\rangle \otimes |\psi_s\rangle \otimes |\psi_{\mathrm{f}}\rangle. \tag{5}$$

$f(z^\alpha)$ is the bosonic part, which does not carry spin nor flavour indices. $|\psi_0\rangle$ comes from the fermionic collective coordinates associated with the center of mass motion. This part of the wave function carries spin, as isovector zero modes λ do, and puts the BPS state into a $N = 2$ multiplet. $|\psi_s\rangle$ also carries spin, and exists only when $n_m \geq 2$ since it corresponds to fermionic coordinates associated with the relative motion. $|\psi_{\mathrm{f}}\rangle$ is due to the isospinor zero modes κ, which do not carry spin but put the states into flavour multiplets. For instance, it is clear from (4) that for $n_m = 1$ the BPS states are in spinorial representation of the flavour symmetry group $\mathrm{Spin}(2N_f)$. In general, one can obtain constraints relating the flavour representations and the electric charge carried by the states using the form (5) of the wave function and the fact that the isospinor zero modes pick up a minus sign under a 2π electric rotation. These constraints were listed e.g. in [7]; they limit the possible decay reactions across curves of marginal stability and thus provide consistency checks of the predicted strong and weak coupling spectra [7].

Let me close this discussion of the quantum numbers with a remark. In the $N = 2$ theory, there are only two complex zero modes carrying spin for $n_m = 1$. These are the spin 1/2 isovector zero modes. We thus see that a monopole of unit magnetic charge cannot carry spin greater that 1/2 (they lie in standard $N = 2$ matter hypermultiplet) and by the way cannot be the dual of the W bosons (in the $N = 4$ theory there are twice as many isovector zero modes and this problem disappears). However, for $n_m \geq 2$, we have additional zero modes carrying spin, which are included in $|\psi_s\rangle$, and thus these states may be dual to the W. This is what one expects to occur in the $N = 2$ theory with four flavours.

3. The non perturbative approach

Now I wish to present the method used in [6, 7], where the BPS spectra of the asymptotically free theories ($0 \leq N_f \leq 3$) were rigorously computed, both at weak and strong coupling. This leads in particular to the first description of the decay reactions across the curves of marginal sta-

bility at strong coupling (an alternative approach from string theory has appeared since then [18]). The method is non perturbative in nature and uses constraints coming from the global analytic structure of the Coulomb branch of the theories. The remarkable, and unexpected, result that emerges is the following: there is one and only one BPS spectrum compatible with the global low energy structure of the theories. Whether this is a very general statement, or is limited to the special cases studied so far, is an open question.

3.1. General analysis

In the theories under study, the scalar potential has flat directions which cannot be lifted quantum mechanically due to tight constraints coming from supersymmetry [19]. These flat directions generate a moduli space which has a Coulomb branch along which the gauge group SU(2) is spontaneously broken down to U(1) by the Higgs expectation value $\langle\phi\rangle = a\sigma_3$. A good, gauge invariant coordinate along the Coulomb branch is $u = \langle \operatorname{tr}\phi^2\rangle$. The low energy, wilsonian, effective action can be expressed in terms of a single holomorphic function $\mathcal{F}$ (the prepotential) because of $N = 2$ supersymmetry as

$$\mathcal{L}_{\text{eff}} = \frac{1}{8\pi}\,\Im m\left[2\int d^2\theta d^2\overline{\theta}\,\mathcal{F}'(A)\overline{A} + \int d^2\theta\,\mathcal{F}''(A)W^2\right]. \qquad (6)$$

(W, A) is the $N = 2$ abelian massless vector multiplet. One introduces traditionally $a_D = \frac{1}{2}\mathcal{F}'(a)$ and the coupling constant is then $\tau(a) = \mathrm{d}a_D/\mathrm{d}a = \theta/\pi + \mathrm{i}8\pi/g^2$, θ representing the low energy θ angle and g the gauge coupling constant.

For concreteness, I will exclusively study the $N_f = 1$ theory in the following. It exhibits all the main features of the other asymptotically free theories. Asymptotic freedom is used here to deduce the form of the gauge coupling $g(a)$ when a goes to infinity from the perturbative β function $\beta = -3g^3/(16\pi^2)$:

$$\frac{1}{g^2} = \frac{3}{8\pi^2}\,\ln\frac{|a|}{\Lambda}. \qquad (7)$$

Holomorphy, and the fact that $a(u) \sim \sqrt{u/2}$, then yield

$$a_D(u) \sim \frac{3\mathrm{i}}{2\pi}\,\sqrt{u/2}\,\ln\frac{u}{\Lambda^2}. \qquad (8)$$

$\Lambda \propto a\exp(-8\pi^2/3g^2)$ is the dynamically generated scale of the theory. As we will see below, it has a clear physical signification since it gives the typical scale at which singularities appear on the Coulomb branch. The n instanton corrections to the perturbative results are proportional to $\exp(-8\pi^2 n/g^2) \propto \Lambda^{3n}$. Because of a flavour parity symmetry existing in the theory, only even numbers of instantons contribute (see e.g. [2]) and we expect to have

$$a(u) = \sqrt{u/2}\left[1 + \sum_{k=1}^{\infty} a_k\left(\frac{\Lambda^2}{u}\right)^{3k}\right]$$
$$a_D(u) = \frac{3\mathrm{i}}{2\pi}\sqrt{u/2}\ln\frac{u}{\Lambda^2} + \sqrt{u}\sum_{k=1}^{\infty} a_{Dk}\left(\frac{\Lambda^2}{u}\right)^{3k}. \qquad (9)$$

This is a non perturbative, but still semiclassical, formula, since it comes from the application of the steepest descent method to the functional integral. In particular, the series have a finite radius of convergence, and we must find an analytic continuation in order to have a global description. How to find this analytic continuation was done by Seiberg and Witten in their celebrated papers [1, 2].

3.2. Results from Seiberg and Witten and general remarks

Using some physical arguments, it was shown in [2] that the low energy effective action of the $N_f = 1$ theory under study has three singularities at strong coupling, due to dyons of unit magnetic charge becoming massless. Probably the most important lesson of [1, 2] was to explain how to compute the asymptotics of a and a_D near such strong-coupling singularities. The idea consists in using electric-magnetic duality rotations to couple locally the soliton becoming massless with the $N = 2$ abelian vector multiplet contained in $\mathcal{L}_{\text{eff}}$. One then uses the fact that an abelian gauge theory is weakly coupled in the infrared to deduce the asymptotics of g using the one-loop β function of the low energy theory. This can also be done directly in the microscopic theory, as was

pointed out recently by the present author [20]. Using these asymptotics, the monodromy matrices around the singular points can be deduced. For one hypermultiplet (n_e,n_m) becoming massless, it is in the conventions of [6, 7]

$$M_{(n_e,n_m)}=\begin{pmatrix}1-n_en_m & n_e^2\\ -n_m^2 & 1+n_en_m\end{pmatrix}. \qquad (10)$$

One important point to realize is that the structure of the monodromy group is extremely constrained by the discrete $\mathbf{Z}_3$ symmetry acting on the Coulomb branch. This discrete symmetry comes from the anomaly free part $\mathbf{Z}_{12}$ of the $\mathrm{U}(1)_{\mathrm{R}}$ symmetry of the classical theory, under which u has charge 4. It imposes that the three singularities must be at the vertices of an equilateral triangle, say at $u_1=e^{\mathrm{i}\pi/3}$, $u_2=-1$ and $u_3=e^{-\mathrm{i}\pi/3}$, choosing $\Lambda\sim 1$. Moreover, if (n_e,n_m) becomes massless at u_3, then (n_e-n_m,n_m) will be massless at u_1 and (n_e-2n_m,n_m) at u_2. As noticed in [7], consistency with the monodromy at infinity

$$M_\infty=\begin{pmatrix}-1 & 3\\ 0 & -1\end{pmatrix} \qquad (11)$$

actually implies that $n_m=\pm 1$. We will choose to have $(0,1)$ massless at $u=u_3$.

To obtain an explicit solution for a_D and a is now a purely mathematical exercise. Using the theory of differential equations, it is not so difficult to find a second order linear differential equation whose solutions have the required monodromies. Nevertheless, the most elegant and general way of doing this [1, 2], keeping in mind that the monodromies are elements of $\mathrm{SL}(2,\mathbf{Z})$, is to find a family of cubic complex curves (tori) parametrized by u, whose modular parameter is determined modulo the modular group at fixed u. a and a_D are then expressed as period integrals, and the differential equations are nothing but the Picard-Fuchs equations associated with them. Then one has to solve these differential equations (or compute directly the periods integrals) to obtain the solution explicitly. The latter allows to obtain the curve of marginal stability, and exhibits the global analytic structure, two crucial ingredients for our purposes.

3.3. Analysis of the explicit solution

The solution, found in [7], reads for $-2\pi/3\leq \arg u\leq 0$

$$\begin{aligned} a(u) &= \sqrt{u/2}\,F\left(-\frac{1}{6},\frac{1}{6},1;-\frac{1}{u^3}\right)\\ a_D(u) &= e^{-2\mathrm{i}\pi/3}\,\frac{\sqrt{2}}{12}\left(u^3+1\right)\\ &\qquad F\left(\frac{5}{6},\frac{5}{6},2;1+u^3\right), \end{aligned} \qquad (12)$$

and we have for all u

$$\begin{aligned}\begin{pmatrix}a_D\\ a\end{pmatrix}\left(e^{\pm 2i\pi/3}u\right) &= e^{\pm i\pi/3}G_{W\pm}\begin{pmatrix}a_D\\ a\end{pmatrix}(u),\\ G_{W\pm} &= \begin{pmatrix}1 & \mp 1\\ 0 & 1\end{pmatrix}. \end{aligned} \qquad (13)$$

The latter equation reflects the $\mathbf{Z}_3$ symmetry acting on the Coulomb branch. The mass of any BPS states (n_e,n_m) can then be computed very explicitly using the formula [1, 2]

$$m=\sqrt{2}\,|an_e-a_Dn_m|. \qquad (14)$$

The curve $\mathcal{C}$ of marginal stability across which the spectrum of BPS states is discontinuous is the locus of the points u where the two dimensional lattice generated by a and a_D collapses to a line:

$$\mathcal{C}=\left\{u\mid \Im m\,\frac{a_D}{a}=0\right\}\cdot \qquad (15)$$

On this curve, states (n_e,n_m) which are ordinarily stable (i.e. n_e and n_m are relatively prime, or the state corresponds to a bound state at threshold like the W bosons $(\pm 2,0)$) may become unstable and “decay” [1, 2, 6, 7]. $\mathcal{C}$ was computed numerically in [7]. It looks like a circle, and contains the three singularities as is clear from (14). It is depicted in the Figure, where the cuts of the functions a_D and a are also represented.

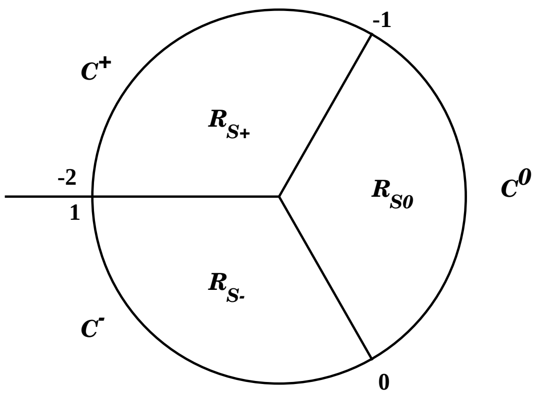

Figure. The curve of marginal stability $\mathcal{C}$ passes through the three cubic roots of -1. It is almost a circle. The numbers 1, 0 -1, -2 indicate the values taken by a_D/a along the curve. The definitions of the various portions of the curve and of the strong coupling region are indicated.

The fact that the cuts separate the strong coupling region R_S (inside the curve) into three parts R_{S+}, R_{S-} and R_{S0} has a crucial physical meaning. To understand this, consider a BPS state (n_e, n_m), say in R_{S0}. Its mass is given by (14). Now, vary u continuously inside the strong coupling region and go for instance in the region R_{S+}. Physically, nothing happens when crossing the cut, since the BPS state remains stable (we do not cross the curve of marginal stability $\mathcal{C}$ in this process). In particular the mass of the state must vary continuously, and thus is given in R_{S+} by $m = \sqrt{2}\,|\tilde{a}n_e - \tilde{a}_D n_m|$ where $\tilde{a}$ and $\tilde{a}_D$ are the analytic continuations in R_{S+} of a and a_D through the cut separating R_{S0} and R_{S+}. The relation between (a_D, a) and $(\tilde{a}_D, \tilde{a})$ is given by the monodromy around $u_1 = e^{\mathrm{i}\pi/3}$. If we insist in using the solution (a_D, a) given by (12, 13) all through the u-plane, we see that one cannot label a BPS state by a unique set of quantum numbers all through the strong coupling region. If (n_e, n_m) is used in R_{S0}, then we must use $(\tilde{n}_e, \tilde{n}_m)$ such that $|a\tilde{n}_e - a_D\tilde{n}_m| = |\tilde{a}n_e - \tilde{a}_D n_m|$ in R_{S+}. This is equivalent to the fact that the $\mathrm{SL}(2, \mathbf{Z})$ bundle over the Coulomb branch, of which (a_D, a) and (n_e, n_m) are sections, is not trivial. The strong coupling monodromies give the transition functions. Explicitly, denoting by p the locally constant section representing a given BPS state, we have [7]

$$\begin{aligned} p \equiv (n_e, n_m) \quad &\text{in} \quad R_{S0} \\ \Leftrightarrow \quad & p \equiv \pm(n_e, n_m + n_e) \text{ in } R_{S-} \\ \Leftrightarrow \quad & p \equiv \pm(2n_e + n_m, -n_e) \text{ in } R_{S+}. \end{aligned} \quad (16)$$

Some remarks concerning these formulas are worthwhile.

First, note that a state becoming massless must exist at strong coupling, since one can cross $\mathcal{C}$ precisely at the point where it is massless and thus stable (it is the only massless charged state at this point). For instance, one can follow continuously the state $(0, 1)$ from weak coupling to strong coupling crossing $\mathcal{C}$ at $u = u_3 = e^{-\mathrm{i}\pi/3}$. Entering in the strong coupling region, one can choose in this case to go either in R_{S-} or in R_{S0}. Since the transition from weak coupling is continuous, the state $(0, 1)$ should be represented by the same quantum numbers in R_{S-} and R_{S0}, and these quantum numbers can be computed semiclassically. Note that this is valid for all the quantum numbers eventually carried by the state. For the electric and magnetic charge, this implies in particular that $(0, 1)$ must represent the same state in R_{S0} and in R_{S-}, which is indeed the case, see (16). More generally, this provides a physical interpretation of the fact that (n_e, n_m) is always an eigenvector of eigenvalue 1 of the monodromy matrix $M_{(n_e,n_m)}$ (10).

Second, note that the transformation rules (16) intimately mix the electric and magnetic quantum numbers. For instance, the magnetic monopole $(n_e = 0, n_m = 1)$ becoming massless at $u = u_3$ will be described by $(n_e = \pm 1, n_m = 0)$ in R_{S+}! This phenomenon nicely illustrates the fact that the distinction between the electric and magnetic quantum numbers is very unclear at strong coupling, unlike at weak coupling where they have a completely different origin.

3.4. The semiclassical spectrum

Let us quit for a moment the strongly coupled physics, and focus on the weak coupling spectrum S_W. Next I will present an argument [6, 7] which allows to completely determine it, in a surpris-

ingly easy way when one has in mind that what we really do is to count zero modes of a supersymmetric hamiltonian, as explained in Section 1, or equivalently to compute the cohomology of very complicated manifolds!

First, the elementary excitations $\pm(1,0)$ (quarks) and $\pm(2,0)$ (W bosons) must be in S_W, as well as the states responsible for the singularities, that is $\pm(1,1)$, $\pm(0,1)$ and $\pm(-1,1)$. Looping around the point at infinity, one can then deduce that all the states $\pm(n_e,1)$ are in S_W, for all integers n_e. Note that this means that electric charge is quantized, which simply reflects that the collective coordinate associated with it is a periodic variable. What I wish to prove now is that no states of magnetic charge greater than or equal to two exist. Suppose the contrary, and let (n_e,n_m) be such a state. Using the monodromy at infinity as above, one deduces immediately that the states (n_e-3kn_m,n_m) are also in S_W for all integers k. Let us look at the particular state $(n_e-3k_0n_m,n_m)$ where k_0 is chosen such that $(n_e-3k_0n_m)/(n_m)\in[-2,1]$. Because a_D/a takes all values in the interval $[-2,1]$ along the curve of marginal stability $\mathcal{C}$, which follows most easily from the monodromy at infinity, there exists a point $u^*\in\mathcal{C}$ such that $(n_e-3k_0n_m)a(u^*)-n_ma_D(u^*)=0$. This means that $(n_e-3k_0n_m,n_m)$ is massless at u^*, which is impossible as the only singularities on the Coulomb branch are those due to the states $n_m=1$ mentioned above. This complete the proof.

3.5. The strong-coupling spectrum

We now have all the necessary ingredients to determine the strong coupling spectrum. We already know that the three states responsible for the singularities are in S_S, as they are stable across the points where they are massless. Actually, there cannot be any other state. Note first that a_D/a varies from 1 to 0 along $\mathcal{C}^-$, from 0 to -1 along $\mathcal{C}^0$, and from -1 to -2 along $\mathcal{C}^+$. Note also that the part $\mathcal{C}^0$ of the curve borders the region R_{S0}, $\mathcal{C}^+$ the region R_{S+}, $\mathcal{C}^-$ the region R_{S-} (see the Figure), and that different quantum numbers must be used to describe a BPS state in these different regions (16). Thus, a state represented by (n_e,n_m) in R_{S0}, $n_m\neq 0$, is never massless if and only if $r=n_e/n_m\notin[-1,0]$, $n_e/(n_m+n_e)=r/(1+r)\notin[0,1]$ and $-(2n_e+n_m)/n_e=-2-1/r\notin[-2,-1]$. But this is impossible from very elementary analysis! The case $n_m=0$ is left to the reader, and we arrive at our main conclusion: in the strong coupling region, only the states responsible for the singularities exist. In particular, no normalizable quantum states correspond to the W bosons or to the quarks in the theory, though they appear as elementary fields in the lagrangian! Note that this fact can be suspected using an independent (heuristic) argument for the W (applied in [21] to the pure gauge theory).

I wish to conclude this subsection by pointing out that we do not use explicitly the global discrete symmetry in the reasoning above. However, the analytic structure is strongly constrained by this symmetry, as already noted above, and what we do is strictly equivalent to the reasoning favoured in [6, 7]. Using explicitly the global discrete symmetry would allow to work in a fixed region, say R_{S0}. It also leads to the funny conclusion that the BPS states must come in multiplets of the symmetry, though it is spontaneously broken, an aspect emphasized in [22].

3.6. Fun with the quantum numbers

In finding two different spectra at weak coupling and strong coupling, we have predicted a set of decay reactions across the curve $\mathcal{C}$ (for a discussion of what these "decay reactions" really are, see [7]). All these decay reactions should be compatible with the conservation of the quantum numbers carried by the BPS states: mass, spin, electric, magnetic and flavour charges. These quantum numbers are unambiguously determined at weak coupling. For instance, the flavour charge F (which is an abelian U(1) charge for $N_f=1$) is $+1/2$ for a state $(2k,1)$ and $-1/2$ for $(2k+1,1)$. At strong coupling, there is a sign ambiguity in n_e, n_m and F because the transition functions (16) are only determined up to a sign. It can be shown that this sign ambiguity is lifted if one wants the decay reactions to be possible. I refer the reader to [7] for more details.

4. Conclusion

We have presented two methods which allow to compute the BPS spectra in $N = 2$ supersymmetric gauge theories. The first method is semiclassical and has lead to interesting results, particularly for the $N = 4$ theory. However, it requires to deal with difficult mathematics, and cannot account, even in principle, for the discontinuity of the spectrum at strong coupling. The second method is non perturbative in nature, and has proved to be very powerful and simple for the asymptotically free theories. It provides a very easy way to compute the semiclassical spectrum, and gives also the full answer at strong coupling. The main challenge for the future will be to extend this method to the conjectured self-dual field theories and try to get here new insights on electric-magnetic duality.

Acknowledgements

I am particularly grateful to the scientific directors of the '96 spring workshop on string theory, gauge theory and quantum gravity at ICTP, namely R. Dijkgraaf, R. Iengo, I. Klebanov, K.S. Narain and S. Randjbar-Daemi, for inviting me to present the material covered in these notes.

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